Chapter 12: Limits and Derivatives

12.1 Estimating Limits Graphically

**Limit** -

\[
\lim_{x \to c} f(x) = L
\]

It is important to understand that a limit is not about what happens at the number that \( x \) is approaching. Instead, a limit is about what happens near or close to that number.

**EXAMPLE 1** Estimate a Limit = \( f(c) \)

Estimate \( \lim_{x \to 2} (-3x + 1) \) using a graph. Support your conjecture using a table of values.

**Guided Practice** (Use graphing calculator)

Estimate each limit using a graph. Support your conjecture using a table of values.

1A. \( \lim_{x \to -3} (1 - 5x) \)  
1B. \( \lim_{x \to 1} (x^2 - 1) \)
Use a graphing calculator to find the limits, if they exist.

\[ \lim_{x \to 7} 2x = \]

\[ \lim_{x \to -8} \frac{x}{x + 4} = \]

\[ \lim_{x \to 11} \sqrt{2x - 6} = \]

\[ \lim_{x \to 2^+} \sqrt{x - 2} = \]

\[ \lim_{x \to 3^-} 5\sqrt{x - 3} = \]

\[ \lim_{x \to \infty} \frac{2}{x - 4} = \]
For a limit to exist

\[
\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)
\]

### Key Concept

#### One-Sided Limits

**Left-Hand Limit**

If the value of \( f(x) \) approaches a unique number \( L_1 \) as \( x \) approaches \( c \) from the left, then

\[
\lim_{x \to c^-} f(x) = L_1, \text{ which is read } \lim_{x \to c^-} f(x)
\]

*The limit of \( f(x) \) as \( x \) approaches \( c \) from the left is \( L_1 \).*

**Right-Hand Limit**

If the value of \( f(x) \) approaches a unique number \( L_2 \) as \( x \) approaches \( c \) from the right, then

\[
\lim_{x \to c^+} f(x) = L_2, \text{ which is read } \lim_{x \to c^+} f(x)
\]

*The limit of \( f(x) \) as \( x \) approaches \( c \) from the right is \( L_2 \).*

### Example 3

**Estimate One-Sided and Two-Sided Limits**

Estimate each one-sided or two-sided limit, if it exists.

a. \( \lim_{x \to 0^-} \frac{|2x|}{x}, \lim_{x \to 0^+} \frac{|2x|}{x}, \text{ and } \lim_{x \to 0^+} \frac{|2x|}{x} \)

### Guided Practice

3A. \( \lim_{x \to 1^-} f(x), \lim_{x \to 1^+} f(x), \text{ and } \lim_{x \to 1} f(x) \),

where \( f(x) = \begin{cases} 
x^3 + 2 & \text{if } x < 1 \\
2x + 1 & \text{if } x \geq 1
\end{cases} \)
EXAMPLE 2  Estimate a Limit ≠ f(c)

Estimate \( \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \) using a graph. Support your conjecture using a table of values.

\[
\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3}
\]

Guided Practice

Estimate each limit using a graph. Support your conjecture using a table of values.

2A. \( \lim_{x \to -2} \frac{x + 2}{x^2 - 4} \)  
2B. \( \lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} \)

EXAMPLE 4  Limits and Unbounded Behavior

Estimate each limit, if it exists.

a. \( \lim_{x \to 4} \frac{1}{(x - 4)^2} \)

b. \( \lim_{x \to 0} \frac{1}{x} \)
Guided Practice

4A. \[ \lim_{x \to 3} \frac{x^2 - 4}{x - 3} \]  
4B. \[ \lim_{x \to 0} \frac{2}{x^4} \]

Example 5  
Limits and Oscillating Behavior

Estimate \( \lim_{x \to 0} \cos \frac{1}{x} \), if it exists.

Guided Practice

Estimate each limit, if it exists.

5A. \[ \lim_{x \to 0} \sin \frac{1}{x} \]  
5B. \[ \lim_{x \to 0} (x^2 \sin x) \]
EXAMPLE 6  
Estimate Limits at Infinity

Estimate each limit, if it exists.

a. \( \lim_{x \to \infty} \frac{1}{x^2} \)

b. \( \lim_{x \to \infty} \left( \frac{3}{x^2} + 2 \right) \)

Guided Practice

6A. \( \lim_{x \to \infty} \left( \frac{1}{x^4} - 3 \right) \)  
6B. \( \lim_{x \to \infty} e^x \)  
6C. \( \lim_{x \to \infty} \sin x \)


**Concept Summary**

**Why Limits at a Point Do Not Exist**

The limit of \( f(x) \) as \( x \) approaches \( c \) does not exist if:

- \( f(x) \) approaches a different value from the left of \( c \) than from the right,
- \( f(x) \) increases or decreases without bound from the left and/or the right of \( c \), or
- \( f(x) \) oscillates between two fixed values.


**HOMEWORK (ODDS only)**

Estimate each limit using a graph. Support your conjecture using a table of values. (Examples 1 and 2)

1. \( \lim_{{x \to 3}} (4x - 10) \)
2. \( \lim_{{x \to 2}} \left( \frac{1}{x^5} - 2x^3 + 3x^2 \right) \)
3. \( \lim_{{x \to -2}} (x^2 + 2x - 15) \)
4. \( \lim_{{x \to -2}} \frac{x^3 + 8}{x^3 - 4} \)
5. \( \lim_{{x \to 3}} (2x^3 - 10x + 1) \)
6. \( \lim_{{x \to 0}} \frac{x \cos x}{x^2 + x} \)
7. \( \lim_{{x \to 0}} [5(\cos^2 x - \cos x)] \)
8. \( \lim_{{x \to 4}} \frac{x - 4}{\sqrt{x} - 2} \)
9. \( \lim_{{x \to \infty}} (x + \sin x) \)
10. \( \lim_{{x \to -5}} \frac{x^2 + x - 20}{x + 5} \)

For each function below, estimate each limit if it exists. (Examples 1–4)

29. \( \lim_{{x \to 0}} f(x) \)
30. \( \lim_{{x \to 0}} f(x) \)
31. \( \lim_{{x \to 0}} g(x) \)
32. \( \lim_{{x \to 0}} g(x) \)
Estimate each limit, if it exists. (Examples 4–6)

33. \( \lim_{x \to 4} \frac{-17}{x^2 + 8x + 16} \)
34. \( \lim_{x \to 5} \frac{x^2}{x^2 - 10x + 25} \)
35. \( \lim_{x \to 4} \frac{|x|}{x - 4} \)
36. \( \lim_{x \to -\infty} x^{2x} - 5 \)
37. \( \lim_{x \to -6} \frac{5}{(x - 6)^2} \)
38. \( \lim_{x \to -\infty} (x^3 - 7x^4 - 4x + 1) \)

For the function below, estimate each limit if it exists.

53. \( \lim_{x \to 0^-} f(x) \)
54. \( \lim_{x \to 0^+} f(x) \)
55. \( \lim_{x \to 0} f(x) \)
56. \( \lim_{x \to 2^-} f(x) \)
57. \( \lim_{x \to 2^+} f(x) \)
58. \( \lim_{x \to 1} f(x) \)

H.O.T. Problems Use Higher-Order Thinking Skills

65. ERROR ANALYSIS Will and Kenyi are finding the limit of the function below as \( x \) approaches \(-6\). Will says that the limit is \(-4\). Kenyi disagrees, arguing that the limit is \(3\). Is either of them correct? Explain your reasoning.
Practice

Estimating Limits Graphically

Estimate each one-sided or two-sided limit, if it exists.

1. \[ \lim_{x \to 0} (4 - \sqrt{x}) \]
2. \[ \lim_{x \to 3} \frac{3 - x}{|x - 3|} \]

3. \[ \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \]
4. \[ \lim_{x \to -1} \frac{x + 7}{x^2 + 8x + 7} \]

5. \[ \lim_{x \to -1, x^2 + 8x + 7} \frac{x + 7}{x} \]
6. \[ \lim_{x \to 0} \frac{x^2 + 1}{x^2} \]

Estimate each limit, if it exists.

7. \[ \lim_{x \to -\infty} \frac{-4x^2}{x^2 + 1} \]
8. \[ \lim_{x \to -\infty} \frac{3x - 2}{x - 1} \]

9. \[ \lim_{x \to 0} \frac{\sin 2x}{x} \]
10. \[ \lim_{x \to \infty} e^{3x + 2} \]

11. RATE OF CHANGE A 20-foot pole is leaning against a barn. If the base of the pole is pulled away from the barn at a rate of 3 feet per second, the top of the pole will move down the side of the barn at a rate of \( r(x) = \frac{3x}{\sqrt{400 - x^2}} \) feet per second, where \( x \) is the distance between the base of the pole and the barn. Graph \( r(x) \) to find \( \lim_{x \to 20} r(x) \).

12. POLLUTANTS The cost in millions of dollars for a company to clean up the pollutants created by one of its manufacturing processes is given by \( C = \frac{312x}{100 - x} \), where \( x \) is the number of pollutants and \( 0 \leq x \leq 100 \).

Find \( \lim_{x \to 100^-} C \).
Study Guide and Intervention (continued)

Estimating Limits Graphically

Estimate Limits at Infinity

- If the value of \( f(x) \) approaches a unique number \( L \) as \( x \) increases, then \( \lim_{x \to \infty} f(x) = L_+ \).
- If the value of \( f(x) \) approaches a unique number \( L_- \) as \( x \) decreases, then \( \lim_{x \to -\infty} f(x) = L_- \).

Example: Estimate \( \lim_{x \to \infty} \frac{1}{x + 3} \), if it exists.

Analyse Graphically The graph of \( f(x) = \frac{1}{x + 3} \) suggests that \( \lim_{x \to \infty} \frac{1}{x + 3} = 0 \).

As \( x \) increases, the height of the graph gets closer to 0. The limit indicates a horizontal asymptote at \( y = 0 \).

Support Numerically Make a table of values, choosing \( x \)-values that grow increasingly large.

<table>
<thead>
<tr>
<th>( x )</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.08</td>
<td>0.01</td>
<td>0.001</td>
<td>0.0001</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

The pattern of outputs suggests that as \( x \) grows increasingly larger, \( f(x) \) approaches 0. This supports our graphical analysis.

Exercises

Estimate each limit, if it exists.

1. \( \lim_{x \to \infty} \frac{2x + 1}{x} \)
2. \( \lim_{x \to -\infty} \frac{-3x + 1}{x - 2} \)
3. \( \lim_{x \to \infty} \frac{1}{x^2} \)
4. \( \lim_{x \to \infty} \frac{2x^2 - 5}{3x^2 + 2x} \)
5. \( \lim_{x \to \infty} (\sin x + 2\sin x) \)
6. \( \lim_{x \to -\infty} (2^x + x) \)
7. \( \lim_{x \to \infty} (x \sin x) \)
8. \( \lim_{x \to -\infty} e^{2x} \)
9. \( \lim_{x \to \infty} \cos 2\pi x \)