More Group-Ranking Methods and Paradoxes

Different methods of determining a group ranking can give different results. This fact led the Marquis de Condorcet to propose that a choice that could obtain a majority over every other choice should be ranked first for the group.

Again consider the set of preference schedules used in the previous lesson (see Figure 1.4).

\[
\begin{array}{cccc}
A & B & C & D \\
B & C & D & B \\
C & D & D & C \\
D & A & A & A \\
\end{array}
\]

Total number of voters:
\[8 + 5 + 6 + 7 = 26.\]

**Figure 1.4** Preferences of 26 voters.

Condorcet Method -

To examine these data for a Condorcet winner, compare each choice with every other choice. For example, begin by comparing A with B, then with C, and finally with D. Notice in Figure 1.4 that A is ranked higher than B on 8 schedules and lower on 18. (An easy way to see this is to cover C and D on all the schedules.) Because A cannot obtain a majority against B, A cannot be a Condorcet winner. Therefore, there is no need to check to see if A can beat C or D.
Now consider B. You have already seen that B beats A, so begin by comparing B with C. B is ranked higher than C on $8 + 5 + 7 = 20$ schedules and lower than C on 6.

Now compare B with D. B is ranked higher than D on $8 + 5 + 6 = 19$ schedules and lower than D on 7. Therefore, B has a majority over each of the other choices and so is a Condorcet winner.

Since B is a Condorcet winner, it is unnecessary to make comparisons between C and D. Although all comparisons do not always have to be made, it can be helpful to organize them in a table:

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Although Condorcet's method may sound ideal, it sometimes fails to produce a winner. Consider the set of schedules shown in Figure 1.5.

![Figure 1.5 Preferences of 60 voters.](image)

Notice that A is preferred to B on 40 of the 60 schedules but that A is preferred to C on only 20. Although C is preferred to A on 40 of the 60, C is preferred to B on only 20. Therefore there is no Condorcet winner.

You might expect that if A is preferred to B by a majority of voters and B is preferred to C by a majority of voters, then a majority of voters prefer A to C. But the example shows that this need not be the case.

In other mathematics classes you have learned that many relationships are transitive. The relation "greater than" (>, for example, is transitive because if \( a > b \) and \( b > c \), then \( a > c \).

You have just seen that group-ranking methods may violate the transitive property. Because this intransitivity seems contrary to intuition, it is known as a paradox. This particular paradox is sometimes referred to as the Condorcet paradox. There are other paradoxes that can occur with group-ranking methods, as you will see in this lesson's exercises.
Example

Ten representatives of the language clubs at Central High School are meeting to select a location for the clubs' annual joint dinner. The committee must choose among a Chinese, French, Italian, or Mexican restaurant (see Figure 1.6).

![Diagram of preference rankings](image)

**Figure 1.6** Preferences of 10 students.

Find the Condorcet winner.
8. A group of voters have the preferences shown in the following figure.

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9 5 7 4 4

a. Use plurality, Borda, runoff, sequential runoff, and Condorcet methods to find winners.
9. Read the news article about the Google search engine.

a. Does the transitive property apply to individual Google voting? That is, if site A casts a Google vote for site B and site B casts a Google vote for site C, then must site A cast a Google vote for site C?

b. Does the transitive property apply to the Google ranking system? That is, if site A ranks higher than site B and site B ranks higher than site C, then must site A rank higher than site C? Explain.

Is Google Page Rank Still Important?

**Search Engine Journal**
October 6, 2004

Since 1998 when Sergey Brin and Larry Page developed the Google search engine, it has relied on the PageRank Algorithm. Google's reasoning behind this is, the higher the number of inbound links 'pointing' to a Website, the more valuable that site is, in which case it would deserve a higher ranking in its search results pages.

If site 'A' links to site 'B', Google calculates this as a 'vote' for site B. The higher the number of votes, the higher the overall value for site 'B'. In a perfect world, this would be true. However, over the years, some site owners and webmasters have abused the system, implementing some 'link farms' and linking to Websites that have little or nothing to do with the overall theme or topic presented in their sites.
Although Arrow's work means that a perfect group-ranking method will never be found, it does not mean that current methods cannot be improved. Recent studies have led some experts to recommend a system called **approval voting**.

In approval voting, you may vote for as many choices as you like, but you do not rank them. You mark all those of which you approve. For example, if there are five choices, you may vote for as few as none or as many as five.

a. Write a soft drink ballot like the one you used in Lesson 1.1. Place an "X" beside each of the soft drinks you find acceptable. At the direction of your instructor, collect ballots from the other members of your class. Count the number of votes for each soft drink and determine a winner.

b. Determine a complete group ranking.

c. Is the approval winner the same as the plurality winner in your class?

d. How does the group ranking in part b compare with the Borda ranking that you found in Lesson 1.1?